Hayponormality and Spectral picture of Toeplitz Operators on Bergman space With Harmonic Polynomial symbols

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ABSTRACT. In this article, the necessary and sufficient conditions for the hyponormality of Toeplitz operator T_φ with a trigonometric polynomial symbols φ on Hardy space are explored. And we use algebraic spectral properties of Toeplitz operator on Hardy space to characterize normal and hyponormal Toeplitz operators with polynomial symbol.Also we shows some new phenomenon in the spectral theory of Toeplitz operator on the Bergman space.

INTRODUCTION

An elegant and useful theorem of C. Cowen [7] characterises the hyponormality of a Toeplitz operator T_φ on the Hardy space H^2 (T) of the unit circle T ⊂ C by properties of the symbol $\varphi \in L^\infty(T)$ This result makes it possible to answer an algebraic question coming from operator theory { namely, is T_φ hyponormal? - by studying the function ϕ itself. In a recent paper [18] of T. Nakazi and K. Takahashi, Cowen's method is carried out to obtain substantial new information about hyponormal Toeplitz operators and their symbols. In the present paper we study the hyponormality of T_ϕ in the cases where ϕ is a trigonometric polynomial $\varphi(e^{i\theta}) = \sum_{-m}^{N} a_n e^{i\theta}$; the goal here is to find conditions on the coefficients a_nthat are necessary and sufficient for T_φ to be hyponormal. This problem is still rather complicated in general; however, in this study we, are able to offer necessary and sufficient conditions for the normality and hyponormality of T_φ in the cases where the Fourier coefficients of φ satisfy certain extremal and symmetry properties.

In 1909 H. Weyl examined the spectra of all compact perturbations A+K of a single hermitian operator A and discovered that $\lambda \in \sigma(A+K)$for every compact operator K if and only if λis not an isolated eigenvalue of finite multiplicity in σ(A). Today this result is known as Weyl's theorem, and it has been extended. from hermitian operators A to hyponormal operators and to Toeplitz operators by L. Coburn [4], and to seminormal operators by S. Berberian [1].

In this paper we determine properties of continuous functions ' that imply that Weyl's theorem holds for all analytic functions of the Toeplitz operatorT_φ.

This analysis entails an interesting new fact, which seems to be absent from the literature, concerning the continuity of the spectrum:

In this study we, show that, when restricted to the linear manifold of all Toeplitz operators, the spectrum is a continuous (set-valued) function at every Toeplitz operator T_φ with quasicontinuous symbolφ. In fact, somewhat more general results are true.

Let φ is trigonometric polynomial $(e^{i\theta}) = \sum_{-M}^{N} \quad a_n e^{i\theta}$, to find conditions on the coefficients a_n that are necessary and sufficient for T_φ to be hyponormal.

Let $L(H)$ and $K(H)$actually restates denote the algebra of bounded linear operators and the ideal of compact operators on a complex Hilbert space H , and let π denote the canonical map$L(H)\rightarrow L(H)/K(H)$. If$T\in L(H)$ is a Fredholm operator that is, $\pi(\mathbf{T})$ is invertible in $L(H)/K(H)$, then ker and ker T are finiteT^*. Dimensional and the index of T is the integer

$ind\ T = \dim\ \ker T - \dim\ \ker T^*$

The subset of $\sigma(T)$ that is stable under compact perturbations is denoted by $w(T)$ and is called the weyl spectrum of T.

The fredholm operators that have index zero are called weyl operators. The essential spectrum $\sigma_e(T)$ and the weyl spectrum $W(T)$ are defined as follows:

$\sigma_e(T)=\{\lambda\in \not\subset : T-\lambda 1$ is not aFredholm operator } .

$w(T)=\{\lambda\in \not\subset : T-\lambda 1$ is not aWeyl operator } .

Clear $\sigma_e(T)\subseteq w(T)\subseteq \sigma(T)$, although unlike σ_e and σ the weyl spectrum of T need not satisfy the spectral mapping theorem .

The Hilbert space $L^2(\mathbf{T})$has canonical orthonormal basis given by the trigonometric functions $e_n(z)=z^n$, for all $n\in Z$.

The Hardy space $H^2(\mathbf{T})$ is the closed linear span of $\{e_n : n = 0,1.....\}$.

An element f is referred to as analytic if $f \in H^2$ and Co-analytic if $f \in L^2 \ominus H^2$.

If P denotes the projection operator $L^2 \to H^2$, then for every $\varphi \in L^\infty(\mathbf{T})$ the operator T_φ on H^2 defined by

$T_\varphi \, g = P(\varphi g)$ for all $g \in H^2$ Is called the Toeplitz operators with symbol φ .

An operator T is said to be hyponnormal if its selfcommultator $(T^*, T] = T^* T - T T^*$ is positive (semidefinte) .

Theorem [1.C.C.Cowen] which characterize the hyponormality of Toeplitz operator requires one two solve acertain functional equation in the unit ball of H^∞ . The spectral variation with in the Manifold M of Toeplitz operators: Let K be set with the Hausdorff metric of all compact subsets of $\mathbb{C}$.The spectrum is a function $\sigma : L(H) \to k$ mapping each operator T to its spectrum $\sigma(T)$.

The function σ is upper – semicontinuous and σ dose have points of discontinuity. Let $\varphi \in L^\infty$, the operator T_φ is point of continuity for the spectral function $\sigma : L \to k$, where L is a subset of $L(H^2)$ consisting of all Toeplitz operators.

The subspace $H^\infty(\mathbf{T}) + C(\mathbf{T})$ is a closed subalgebra of L^∞. And the elements of the closed selfadjoint subalgebra QC, which is defined to be

$$QC = \left(H^\infty(\mathbf{T}) + C(\mathbf{T}) \right) \cap \left(\overline{H^\infty(\mathbf{T}) + C(\mathbf{T})} \right)$$

are called quasicotinuous functions .

The subspace PC is the closure in $L^\infty(\mathbf{T})$ of the set all piecewise continuous function on $(\mathbf{T})$.

Next, we chactrarize Toeplitz operator T_φ with the symbol φ on the Bergman spaces defined by $T_\varphi f = p(\varphi f)$ for f in Bergman space.

And as a fundanmental problem concerning Toeplitz operators is to bdetemine the spectra interms of the properties of their symols.

It is natural to study the spectra of Toeplitz operators with bounded harmonic symbols on the Bergman space.

Let dA denote the Lebesgue measure on the open unit disk D in the complex plane C, normalized so that the measure of the disk D is 1.

The complex space $L^2(D\,;A)$ is a Hilbert space with inner product:

$$\langle f, g \rangle = \int_D \quad f(z)\, g\, \underline{(\,z\,)}\, dA(\,z\,).$$

The Bergman space L^2_α is the set of those functions $L^2(D\,;A)$ in that are analytic D on. The Toeplitz operator T_φ with the symbol φ on the Bergman space is defined by $T_\varphi f = p(\varphi f)$ for f in the Bergman space L^2_α, p is an Orthogonal projection from $L^2(D\,;A)$ onto L^2_α, and $\varphi \in L^\infty(D\,;dA)$.

Notations: The Toeplitz operator $\sigma(T_\varphi)$, $\sigma_p(T_\varphi)$ and $\sigma_e(T_\varphi)$ denote the spectrum and essential spectrum of The Toeplitz operator T_φ, respectively.

Let N denote the set of nonnegative integers. There is little characterization for the Topological structure of the spectrum of the Toeplitz operator with

A bounded harmonic symbol, even if the symbol is the Harmonic function $\underline{z} + p$ for an analytic polynomial p.

Keywords: Toeplitz operator, Hyponormal, Hardy space, Bergman space,Harmonic polynomial, spectra , spectrum.

METHODOLOGY

In this study, we use algebraic spectral properties of Toeplitz operator on Hardy space to characterize normal and hyponormal Toeplitz operators with polynomial symbol that is Let φ is a trigonometric polynomial of the from $\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N} a_n e^{in\theta}$, where a_{-m} and a_N are nonzero ,then , T_φ is hyponormal when $m \le N$ and $|a_{-m}| \le |a_N|$. Let $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n e^{in\theta}$, where $a_N \ne 0$, and if $c_0, c_1, \ldots c_{N-1} \in \not{C}$ are obtained from the coefficients of φ by solving the recurrence relation: For φ be the trigonometric polynomial $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n\, e^{in\varphi}$, where $a_N \ne 0$, and let $k \in H^\infty$ satisfies $\varphi - k\overline{\varphi} \in H^\infty$, then k necessarily satisfies

$$k\sum_{n=1}^{N}\overline{a}_n \; e^{-in\theta} - \sum_{n-`}^{N} a_{-n} - e^{-in\theta} \in H^\infty$$

The computation of Fourier coefficients $\hat{k}(0),\ldots,\hat{k}(N-1)$ of k is: $\hat{k}(n)=c_n$, for n=0,1, …,N-1 ,

Where $c_0, c_1, \cdots, c_{N-1}$ are determined uniquely from the coefficients of φ by the recurrence relation $c_0 = \dfrac{a_{-N}}{\overline{a}_N}$

$c_n\left(\overline{a}_N\right)^{-1}\left(a_{-N+n} - \sum_{j=0}^{n-1} C_j\, \overline{a_{N-n+j}}\right)$, for n=1, … , N-1 then the Toeplitz operator T_φ is hyponormal when

$$\sum_{j=0}^{N-1}\left|c_j\right| \le 1$$

Also use spectral theory of Toeplitz operators with Harmonic polynomial symbol on the Bergman space to show some new phenomenon.

Let p be a function in $H^{\infty} \cap C\left(\underline{D}\right)$.Then the Toeplitz operator $T_{\underline{z}+p}$ is invertible on the Bergman space L_{α}^{2} if and only if the following two conditions hold:

(i) $1 + zp$ has no zeros on the unit circle ∂D ;

(ii) $1 + zp$ has exactly one simple zero z_0 in the open disk D which satisfies that

$$z_0^2\, \acute{p}(z_0) - \frac{n+2}{n+1} \neq 0$$

For any nonnegative integer n.

Now we show that the spectrum of Toeplitz operator $T_{\underline{z}+p}$ is connected for every quadratic polynomial p .

LITERATURE REVIEW

Here a few essential fact concerning Toeplitz operator with continuous symbols need to begin with, using [8] R,G.Douglas The Hilbert space $L^2(T)$ has a canonical orthonormal basis given by the trigonometric function $e_n(z) = z^n\, for\, all\, n \in Z$, and the Hilbert space $H^2(T)$ is the closed linear span of $\{e_n : n = 0,1, \dots\}$.

An element $f \in L^2$ is referred to analytic if $f \in H^2$ and coanalytic if $f \in L^2 \ominus H^2$ .If p denoted the projection operator $L^2 \rightarrow H^2$, then for every $\varphi \in L^{\infty}(T)$, the operator $T_{\varphi}\, on\, H^2$ defined by

$$T_{\varphi}\, g = p(\varphi g)\, for\, all\, g \in H^2 \qquad (6)$$

Is called the Toeplitz operator with symbol φ . Every Toeplitz operator has connected spectrum and essential spectrum, and by [4] L.A.Coburn,

$$\sigma\left(T_{\varphi}\right) = w\left(T_{\varphi}\right)$$

The set $C\,(T)$ of all continuous complex-valued functions on the unit circle T and $H^{\infty}(T) = L^{\infty} \cap H^2$ are Banach algebras, and it is well - known that every Toeplitz operator with symbol $T_{\varphi} \in H^{\infty}$ is subnormal .The $C^* - algebra\, V$ generated by all Toeplitz operates T_{φ} with $\varphi \in C\,(T)$ has an important property Which is very useful for spectral theory: the commutator ideal of V is the ideal $K(H^2)$ of compact Operators on H^2 . As $C\,(T)$ and $V\,/\,K(H^2)$ are $* -isomorphic\, C^* - algebra$, then for every $\varphi \in C\,(T)$,

$(\, i\,)\, T_{\varphi}$ is a Fredholm operator if and only if φ is invertible ;

$(\, ii\,)\, indT_{\varphi} = -wn\,(\varphi)$,

$(\, iii)\, \sigma_e\left(T_{\varphi}\right) = \varphi\,(T)$,

Where $wn\,(\varphi)$ denotes the winding number of φ with respect to the origin. Finally, we make note that if we make note that if We make note that if $\varphi \in C\,(T)\, and\, if\, f$ is an analytic function defined on an

open set containing $\sigma_e(T_\varphi)$, $then\ f \circ \varphi \in C(T)\ and\ f\ (T_\varphi)$,
Is well-defined by the analytic functional calculus.
It is known that the Weyl spectra of every hermitian operator and every normal operator consist precisely of all points in the spectra except the isolated eigenvalues of finite geometric multiplicity. “Weyl’s theorem for an operator” was first introduced by Coburn [34] in 1966, which says that the complement in the spectrum of the Weyl spectrum coincides with the isolated points of the spectrum which are eigenvalues of finite geometric multiplicity. Moreover, Coburn showed that Weyl’s theorem holds for all hyponormal operators and Hardy-Toeplitz operators [34]. Weyl type theorems with respect to isolated points of the spectrum of an operator were investigated for many cases and many classes of operators. Based on the characterizations for the spectra of Toeplitz operators in Theorems 2.4 and 4.1, we show in Theorem 5.2 that the Bergman-Toeplitz operator $T_{\underline{z}+q}$ satisfies Weyl’s theorem, where q is an arbitrary function in the disk algebra in $H^\infty \cap C\ (\underline{D})$.
Aim: This paper aims to investigate the conditions under which Toeplitz operator with symbols exhibit hyponormality on Hardy space by using algebraic spectra
Properties of it and also Investigate the structure of the spectrum of the Toeplitz operator on Bergman space.

1. Necessary and sufficient conditions for Hyponormality with triagonometric polynomial symbols on Hardy space

Theorem 1.1. If f is an analytic function on an open set containing $\sigma(T)$, then

$$w(f\ (T\ )\ ) \subseteq f\left(w\left(T\right)\right) \qquad (`1)$$

But if T hyponormal , then

$$w(f\left(T\right)) = f\ \left(w\left(T\right)\right) \qquad (2)$$

Remark 1.2. Every Toeplitz operator has connected spectrum and essential spectrum, and

$$\sigma\left(T_\varphi\right) = w\left(T_\varphi\right) \qquad (3)$$

Remark 1.3. The sets $C(\mathbf{T})$ of all continuous complex-valued functions on the unit circle $\mathbf{T}$ and $H^\infty(\mathbf{T}) = L^\infty \cap H^2$ are Banach algebras.

Theorem 1.4. Every Toeplitz operator with symbol $\varphi \in H^\infty$ is subnormal.

Theorem 1.5. The commutator ideal of the C^* - algebra V is the ideal $K\left(H^2\right)$ of compact operators on H^2 .

Theorem 1.6. Let $C(\mathbf{T})$ and $V/K\left(H^2\right)$ are * - isomorphic C^* - algebras, then for every $\varphi \in C(\mathbf{T})$,

T_φ is Fredhdm operator if and only φ is invertible (4)

$ind\ T_{\varphi} = -wn(\varphi)$, (5)

$\sigma_e(T_{\varphi}) = \varphi(\mathbf{T})$ (6)

Where $wn(\varphi)$ denotes the winding number of φ with respect to the origin.

If $\varphi \in C(\mathbf{T})$ and if f is an analytic function defined on an open set containing $\sigma(T_{\varphi})$, them $f\, o\, \varphi \in C(\mathbf{T})$ and $f(T_{\varphi})$ is well defined by the analytic functional calculus.

Remark 1.7. Let φ be the trigonometric polynomial $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} a_n\ e^{in\varphi}$, where $a_N \neq 0$, and let $k \in H^{\infty}$ satisfies $\varphi - k\overline{\varphi} \in H^{\infty}$, then k necessarily satisfies

$$k\sum_{n=1}^{N} \overline{a}_n\ e^{-in\theta} - \sum_{n-`}^{N} a_{-n} - e^{-in\theta} \in H^{\infty} \tag{7}$$

(i) The computation of Fourier coefficients $\hat{k}(0), ..., \hat{k}(N-1)$ of k is: $\hat{k}(n) = c_n$, for n=0,1, …,N-1 ,

Where $c_0, c_1, \cdots, c_{N-1}$ are determined uniquely from the coefficients of φ by the recurrence relation $c_0 = \frac{a_{-N}}{\overline{a}_N}$

$$c_n(\overline{a}_N)^{-1}\left(a_{-N+n} - \sum_{j=0}^{n-1} C_j\, \overline{a_{N-n+j}}\right) \text{,for n=1, … , N-1} \qquad (8)$$

(ii) Therefore if $k_1, k_2 \in \mathcal{E}(\varphi)$, then $c_n = \hat{k}_1(n) = \hat{k}_2(n)$ for all $n = 0,1,..., N-1$ and $k_p(z) = \sum_{j=0}^{N-1} c_j Z^j$, the unique (analytic) polynomial of degree less than N satisfying $\varphi - k\overline{\varphi} \in H^{\infty}$.

(iii) Conversely, if k_p is the polynomial $k_p(z) = \sum_{j=0}^{N-1} c_j\, z^j$, where $c_0, c_1, ...c_{N-1}$ are determined from the recurrence relation (8) , then for every integer $n > 0$, the Fourier coefficients $\varphi - k\overline{\varphi}(-n)$ of $\varphi - k\overline{\varphi}$ satisfy

$$\varphi - k\overline{\varphi} = a_{-n} - \sum_{j=0}^{N-n} c_j\, \overline{a_{n+j}} = \left(a_{-n} - \sum_{j=0}^{N-n-1} c_j\, \overline{a_{n+j}}\right) - c_{N-n}\, \overline{a}_N = 0$$,

Which implies that $\varphi - k_p\overline{\varphi} \in H^2$. But since $\varphi - k_p\overline{\varphi}$ is a polynomial, it follows that $\varphi - k_p\,\overline{\varphi} \in H^{\infty}$.

Remark 1.8. However the relation (8) can always be solved uniquely to produce an analytic polynomial k_p satisfying $\varphi - k_p\bar{\varphi} \in H^{\infty}$, the polynomial k_p need not be contained in the set $\mathcal{E}(\varphi)$, even if $\mathcal{E}(\varphi)$ is known to be nonempty .

Example 1.9. Consider the trigonometric polynomial $\varphi(e^{i\theta}) = e^{-i2\theta} + 2e^{-i\theta} + e^{i\theta} + 2e^{i2\theta}$.

Solving the recurrence relation (8) produces the polynomial $k_p(z) = \frac{1}{2} + \frac{3}{4}z$ which has norm $\|k_p\|_{\infty} = \frac{5}{4} > 1$ making k_p ineligible for membership in $\mathcal{E}(\varphi)$.

On the other hand, a straight forward calculation show that the linear fractional transformation $b(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z}$ satisfies $\varphi - b\bar{\varphi} \in H^{\infty}$, as b maps the unit circle onto itself , b has norm $\|b\|_{\infty} = 1$. Thus $b \in \mathcal{E}(\varphi)$ and so T_{φ} is hyponormal. And Fourier series of b , namely

$$b(e^{i\theta}) \sim \frac{1}{2} + \frac{3}{4}e^{i\theta} - \frac{3}{2}\sum_{j=2}^{\infty}\left(-\frac{1}{2}\right)^n e^{in\theta} = k_p(e^{i\theta}) + h(e^{i\theta}),$$

Converges uniformly on $(\mathbf{T})$ to b and b is finite Blaschke product.

Proof. $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}, a_N \neq 0, c_0, c_1, \ldots, c_{N-1} = \quad c_o = \frac{a_{-N}}{a_N}$

$$c_n = (\bar{a}_N)^{-1}\left(a_{-N+N}\sum_{i=n}^{n-1} c_j \bar{a}_{N-n+j}\right), n = 1, \ldots, N-1$$

$$k_p(z) = \sum_{j=0}^{N-1} c_j z^j = c_0, c_1 \ldots c_{N-1},$$

$$a_{-N}e^{-iN\theta} = e^{-iz\theta} \Rightarrow a_{-N} = \frac{e^{-2i\theta}}{e^{-iN\theta}}, \quad a_N e^{iN\theta} = e^{i2\theta} \Rightarrow a_N = \frac{2e^{i2\theta}}{e^{iN\theta}}, \quad \bar{a}_N = \frac{e^{-i2\theta}}{e^{-iN\theta}}$$

$$c_0 = \frac{a_{-N}}{\bar{a}_N} = \frac{e^{-i2\theta}}{e^{-iN\theta}} \times \frac{1}{2}\frac{e^{-iN\theta}}{e^{-i2\theta}} = \frac{1}{2},$$

$$c_1 = (\bar{a}_N)^{-1}(a_{-N+1} - c_0\bar{a}_{N-1})$$

$$a_{-N+1}e^{i(-N+1)\theta} = 2e^{-i\theta} \Rightarrow a_{-N+1} = \frac{2e^{-i\theta}}{e^{i(-N+1)\theta}}$$

$$a_{-N-1}e^{i(-N+1)\theta} = e^{i\theta} \Rightarrow a_{-N-1} = \frac{e^{i\theta}}{e^{i(-N+1)\theta}}$$

$$c_1 = \frac{1}{2}\frac{e^{i2\theta}}{e^{-iN\theta}}\left[\frac{2e^{-i\theta}}{e^{i(-N+1)\theta}} - \frac{1}{2}\frac{e^{-i\theta}}{e^{i(-N+1)\theta}}\right] = \frac{3}{4}z \quad (z = e^{i\theta})$$

$k_p = c_0 + c_1 z = \frac{1}{2} + \frac{3}{4} z$. satisfying $\varphi - k_p \bar{\varphi} \in H^\infty$, where

$\varphi\left(e^{i\theta}\right) = e^{-i2\theta} + 2e^{-i\theta} + e^{i\theta} + 2e^{i2\theta}$

Theorem 1.10. suppose that φ is a trigonometric polynomial of the from $\varphi\left(e^{i\theta}\right) = \sum_{n=-m}^{N} a_n e^{in\theta}$, where a_{-m} and a_N are nonzero . If T_φ is hyponormal , then $m \le N$ and $|a_{-m}| \le |a_N|$.

Proof . Suppose T_φ is hyponormal, then φ is trigonometric polynomial under certain assumption about the conefficients $\varphi\left(e^{i\theta}\right) = \sum_{n=-m}^{N} a_n e^{in\theta}$ where $|a_N| \neq 0$, let k salisfies $\varphi - k_p \bar{\varphi} \in H^\infty$ then necessarily salisfies (7) , then from (8) $-m \le N$.

$$\varphi\left(e^{i\theta}\right) = \sum_{n=-m}^{N} a_n e^{in\theta} = a_{-m} e^{-im\theta} + a_{-m-1} e^{-i(m+1)\theta} + a_{-m-2} e^{-i(m+2)\theta} + \ldots + a_N e^{iN\theta} a_{-m}$$

Since a_{-m} and a_N are nonzero, let $c_0, \cdots, c_{N-1}$ be the solution of (8) because $|a_N| \neq 0$, we have $|c_{N-m}| = |a_{-m}| / |a_N|$, then there is a function $k \in \varepsilon(\varphi)$ such that $k(N-m) = c_{N-m}$ thus $1 \ge \|k\|_\infty \ge |c_{n-m}| = |a_{-m}| / |a_N|$ which implies that $|a_{-m}| \le |a_N|$.

Proposition 1.11. If $\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n e^{in\theta}$, where $a_N \neq 0$, and if $c_0, c_1, \ldots c_{N-1} \in \mathbb{C}$ are obtained from the coefficients of φ by solving the recurrence relation (8) then the Toeplitz operator T_φ is hyponormal when

$$\sum_{j=0}^{N-1} |c_j| \le 1 \tag{9}$$

Proof. $c_0 = \frac{a_{-N}}{\overline{a_N}}, c_n = \left(\bar{a}_N\right)^{-1} \left(a_{-N+n} - \sum_{j=0}^{n-1} c_j \overline{a_{N-n+j}} \right) n = 1, \ldots, N-1$

$\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n e^{in\theta}$, $k_p(z) = \sum_{j=0}^{N-1} c_j z^j$ satisfies $\varphi - k_p \bar{\varphi} \in H^\infty$ from that $\|k_p\|_\infty \le 1$, then $\|k_p\|_\infty \le \sum_{j=0}^{N-1} |c_j| \le 1$, then $k_p(z) \in \varepsilon(\varphi)$ and so from the Cowen's Theorem T_φ is hyponormal .

Remark 1.12. If $\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n e^{in\theta}$ where $|a_j| \le |a_N|$, for all $j = 2, \ldots, N-1$, then from the recurrence relation (12) we have $\sum_{j=0}^{N-1} |c_j| \le |c_0| + |a_N|^{-2} \sum_{n=1}^{N-1} 2^{n-1} |D_n|$, Where $D_n = \det\begin{pmatrix} a_{-n} & a_{-N} \\ \bar{a}_n & \bar{a}_N \end{pmatrix}$. Therefore if

$$\sum_{n=1}^{N-1} 2^{n-1} |D_n| + |a_{-N}\ a_N| \le |a_N|^2 , \qquad (10)$$

Then by proposition 1.11. , T_φ is hyponormal . Because the left – hand side of (10) depend on $\overline{a_{-N}}$ and a_N and the right – hand side depends on $|a_N|^2$, it follows that T_φ is hyponormal whenever $|a_N|$ is sufficiently large . In particular , the Toeplitz operator with symbol $\varphi + \lambda e^{iN\theta}$ is hyponoarmal whenever $\lambda \in \mathbb{C}$ is such that

$$|\lambda| \ge \sum_{n=1}^{N-1} 2^{n-1}\left(|a_{-n}| + |a_n|\right) + |a_{-N}| + |a_N|$$

Proof. Let $\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n e^{in\theta}$ and $|a_j| \le |a_N|, j = 2, ..., N-1$ from (8)

$$\frac{|a_{-N}|}{|\bar{a}_N|} + |a_N|^2 \sum_{n=1}^{N-2} 2^{n-1} |D_n|$$

$$|D_n| = \left(a_{-n}\bar{a}_N - a_{-N}\bar{a}_n\right)$$

$$c_0 = \frac{a_{-N}}{\bar{a}_N} \qquad (1)$$

$$c_1 = (\bar{a}_N)^{-1}\left(a_{-N+1} - c_0 a_{N-1}\right) \qquad (2) \qquad \to c_1 = (\bar{a}_N)^{-1}\left(a_{-N+1} - \frac{a_{-N}}{\bar{a}_N} a_{N-1}\right)$$

$$c_2 = (\bar{a}_N)^{-1}\left(a_{-N+2} - c_1 a_{N-2+1}\right) \qquad (3) \qquad \to c_2 = (\bar{a}_N)^{-1}\left(a_{-N+2} - c_1 a_{N-2+1}\right)$$

$$c_3 = (\bar{a}_N)^{-1}\left(a_{-N+3} - c_2 a_{N-3+2}\right) \qquad (4) \qquad \to c_3 = (\bar{a}_N)^{-1}\left(a_{-N+3} - c_2 a_{N-3+2}\right)$$

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$$c_n = (\bar{a}_N)^{-1}\left(a_{-N+n} - \sum_{j=0}^{n-1} c_j a_{N-n+j}\right) \qquad (5)$$

From (5) $\sum_{j=0}^{n-1} |c_j| \le 1$

Remark 1.13. If $a_{-N} = \cdots = a_{-2} = 0$, then the solution to the recurrence relation (12) is $c_0 = \cdots = c_{N-2} = 0$ and $c_{N-1} = a_{-1} / \overline{a_N}$, thus the analytic polynomial . $k_p \in H^\infty$ is $k_p(z) = \left(a_{-1} | a_N\right) z^{N-1}$. Therefore the norm of every $k \in H^\infty$ that satisfies $\varphi - k\bar{\varphi} \in H^\infty$ is such that

$\| k \|_\infty \geq \left| \dfrac{a_{-1}}{\bar{a}_n} \right| = \| k_p \|_\infty$.

Therefore, T_φ is hyponormal if and only if $|a_{-1}| \leq |a_N|$.

The following theorem and corollary concern the extremal cases : $|a_{-m}| = |a_N| \neq 0$

Proof. $a_{-N} = \cdots = a_{-2} = 0$. then the solution is $c_0 = \cdots = c_{N-2} = 0$ $c_{N-1} = \dfrac{a_{-1}}{a_N}$, thus $k_p \in H^\infty = \sum_{j=0}^{N-1} c_j z^j$, $c_0 = \dfrac{a_{-N}}{\bar{a}_N}$ since $c_{N-m} = \left| \dfrac{a_{-m}}{\bar{a}_N} \right|$ implies that $c_{N-1} = \left| \dfrac{a_{-1}}{\bar{a}_N} \right|, c_{N-2} = \left| \dfrac{a_{-N-2}}{\bar{a}_N} \right|$ implies $\sum_{j=0}^{N-2} |c_j| \leq 1$ from the proposition (1.2.21) and $\| k_p \|_\infty \leq \sum_{j=0}^{N-1} |c_j| \leq 1$, since , $1 \geq \| k \|_\infty \geq \| c_{N-n} \|$ thus $\| k \|_\infty \geq \left| \dfrac{a_{-1}}{\bar{a}_N} \right| = \| k_p \|_\infty$.

Therefor , T_φ is hyponormal if and only if $|a_{-1}| \leq |a_N|$.

Theorem 1.14. There exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$ of degree equal to the rank of $[T_\varphi^*, T_\varphi]$.

Theorem 1.15. Suppose that $\varphi(e^{i\theta}) = \sum_{n=m}^{N} a_n e^{in\theta}$, where $m \leq N$ and $|a_{-m}| \leq |a_N| \neq 0$, and let $\mathcal{E}(\varphi) \subset H^\infty$ be the subset of all $k \in H^\infty$ for which $\| k \|_\infty \leq 1$ and $\varphi - k\bar{\varphi} \in H^\infty$. The following statements are equivalent .

(i) The Toeplitz operator T_φ is hyponormal .

(ii) For all $k = 1, \ldots, N-1$, $\det \begin{pmatrix} a_{-(m-k)} & a_{-m} \\ \overline{a_{N-m+1}} & \overline{a_N} \end{pmatrix} = 0$.

(iii) The following equation in $\mathbb{C}^m$ holds :

$$\overline{a_N} \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \overline{a_{N-m+1}} \\ \overline{a_{N-m+2}} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix} \qquad (11)$$

(iv) $\mathcal{E}(\varphi) = \left\{ a_{-m} \left(\overline{a_N} \right)^{-1} z^{N-m} \right\}$.

Moreover, if T_φ is hyponormal , then the rank of $[T_\varphi^*, T_\varphi]$ is $N - m$.

Proof . Let $c_0,\cdots,c_{N-1}$ be the solution to (8) ; because $|a_{-m}|=|a_N|\neq 0$,we have $|c_{N-m}|=1$. Note that if $m<N$, then $c_0=\cdots=c_{N-m-1}=0$. If a function $k\in H^\infty$ satisfies $\varphi-k\bar{\varphi}\in H^\infty$, then the Fourier series expansion of k is

$$k=\sum_{j=0}^{N-1} c_j e^{ij\theta}+\sum_{n=N}^{\infty} b_n e^{in\theta}$$ for some set of $b_n\in\mathbb{C}$.

From fact $\|k\|_\infty\geq\|k\|_2$ we have $\|k\|_\infty\geq|c_{N-m}|=1$; if for some $j>(N-m)$ or $n\geq N$ there is a nonzero Fourier coefficient c_j or b_n of k , then

$$\|k\|_\infty\geq\max\left\{\sqrt{|c_{N-m}|^2+|c_j|^2}\,,\sqrt{|c_{N-m}|^2+|b_n|^2}\right\}>1.$$

Thus $\|k\|_\infty=1$ if and only if c_{N-m} is the only nonzero Fourier coefficient of k . Therefore $\mathcal{E}(\varphi)$ can have at most one element: namely $c_{N-m}z^{N-m}$ Hence , statements (i) and (iv) are equivalent . Now statement (i) and (ii) are equivalene; clearly (ii) and (iii) are exact same statement. Suppose that T_φ is hyponormal . Then there exists $k\in\mathcal{E}(\varphi)$ and $k(z)=c_{N-m}z^{N-m}$. Hence , for every $k=1,\cdots,m-1,$

$$0=|c_{N-m+k}|=\left|\frac{1}{\overline{a_N}}\left(a_{-(m-n)}\right)-c_{N-m}\,\overline{a_{N-k}}\right|=\left|\frac{1}{\overline{a_N}}\right|^2\left|\det\begin{pmatrix}a_{-(m-k)} & a_{-m}\\ \overline{a_{(N-k)}} & \overline{a_N}\end{pmatrix}\right|.$$

Conversely, if $\det\begin{pmatrix}a_{-(m-k)} & a_{-m}\\ \overline{a_{(N-k)}} & \overline{a_N}\end{pmatrix}=0$ for all $k=1,\cdots,N-1$ then

$$|c_{N-m+1}|=\left|\frac{1}{\overline{a_N}}\left(a_{-(m-1)}-c_{N-m}\,\overline{a_{N-1}}\right)\right|=\left|\frac{1}{\bar{a}_N}\right|^2\det\begin{pmatrix}a_{-(m-k)} & a_{-m}\\ \overline{a_{(N-1)}} & \bar{a}_N\end{pmatrix}=0.$$

and hence

$$|c_{N-m+2}|=\left|\frac{1}{a_N}\left(a_{-(m-2)}-c_{N-m}\,\overline{a_{N-2}}-c_{N-m-1}\,\overline{a_{N-1}}\right)\right|=\left|\frac{1}{\overline{a_N}}\right|^2\det\begin{pmatrix}a_{-(m-2)} & a_{-m}\\ \overline{a_{(N-2)}} & \overline{a_N}\end{pmatrix}=0.$$

Inductively, we obtain $c_k=0$ for all k=1,…,N-1 . As $c_0=\cdots=c_{N-m-1}=0$, if $m<N$ and $|c_{N-m}|=1$, we have , that the analytic polynomial $k_p(z)=\sum_{j=0}^{N-1}$ is of the form $k_p(z)=c_{N-m}z^{N-m}$ and therefore $k_p\in\mathcal{E}(\varphi)$. This completes the proof that statements (i) and (ii) are equivalent.

Lastly, if T_φ is hyponormal , then $\mathcal{E}(\varphi)=\left\{\dfrac{a_{-m}}{\overline{a}_N}\ z^{N-m}\right\}$. Because the self commentator $\left[T_\varphi^*, T_\varphi\right]$ has finite rank , ([18] , theorem 10) , there is only one element in $\mathcal{E}(\varphi):b(z)=\dfrac{a_m}{\overline{a}_N}z^{N-m}$, which is a finite Blaschke product of degree N-m .

Corollary 1.16. Suppose that $\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N}a_n e^{in\theta}$, where $m\le N$ and $|a_{-m}|=|a_{-N}|\neq 0$ and let $\varepsilon(\varphi)\subset H^\infty$ be the subset of all $k\in H^\infty$ for which $\|k\|_\infty\le 1$ and $\varphi-k\overline{\varphi}\in H^\infty$.the following statements are equivalent :

(i)The Toeplitz operator T_φ is hyponormal .

(ii) For all $k=1,\dots,N-1$, $\det\begin{pmatrix}\dfrac{a_{-(m-k)}}{a_{N-m+1}} & \dfrac{a_{-m}}{a_N}\end{pmatrix}=0$.

Proof. Suppose T_φ is hyponormal from Theorem 1.12. the analytic polynomial (ii) holds for all $k=1,\dots,N-1$. For backward implication since $|\overline{a}_N|\neq 0$ and $|\overline{a}_{-m}|\neq 0$,

$$\det\begin{pmatrix}\dfrac{a_{-(m-k)}}{a_{(N-k)}} & \dfrac{a_{-m}}{\overline{a}_N}\end{pmatrix}=a_{-(m-k)}\overline{a}_N-a_{-m}\overline{a}_{N-k}=a_{-m+k}\overline{a}_N-a_{-m}a_{N-k}$$

$$=a_{-m+k}\overline{a}_N-a_{-m+k}\overline{a}=0$$

Then from proposition 1.11.and the remarks 1.12. and 1.13. T_φ is hyponormal

Example 1.17. T_φ is hyponormal with rank-2self commutator rank $\left[T_{\varphi_2}^*, T_{\varphi_2}\right]=2$,
$\varphi_2\left(e^{i\theta}\right)=e^{-i2\theta}+e^{-i\theta}+e^{i3\theta}+e^{i4\theta}$

Prove that T_{φ_2} is hyponormal with rank-2self commutator .

Proof. Let $c_0=c_1=\dots,c_{N-m-1}=0$, $k:1=1,\dots,N-1,\det\begin{pmatrix}\dfrac{a_{-(m-n)}}{a_{(N-n)}} & \dfrac{a_{-m}}{\overline{a}_N}\end{pmatrix}=0$,

$\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N}a_n e^{in\theta}$, $c_0=\dfrac{a_{-m}}{\overline{a}_N}$, $c_{-m-1}=\dfrac{a_{-m-1}}{\overline{a}_N}$, $\dfrac{a_{-m-2}}{\overline{a}_N}=c_{-m-2}\dots c_{-m-(N-1)}=\dfrac{a_{-m-(N-1)}}{\overline{a}_N}$

Then

$$\begin{pmatrix}e^{-i\theta} & e^{-2i\theta}\\ e^{-i3\theta} & e^{-i4\theta}\end{pmatrix}=e^{-5i\theta}-e^{-5i\theta}=0$$

Then T_{φ_2} is hyponormal .

Example 1.18. Applied Theorem 1.15 to show that the Toeplitz operator with symbol

$\varphi\left(e^{i\theta}\right)=e^{-i5\theta}-e^{-i4\theta}+e^{-i2\theta}+e^{-i\theta}+2e^{i2\theta}-2e^{i4\theta}+2e^{i5\theta}$

Whose coefficients satisfy the symmetric relation?

$$\overline{a_N}\begin{pmatrix} a_{-2} \\ a_{-3} \\ \vdots \\ a_{-N} \end{pmatrix}=a_{-N}\begin{pmatrix} \bar{a}_{-2} \\ \bar{a}_{-3} \\ \vdots \\ \bar{a}_{-N} \end{pmatrix}$$

But for which there is no symmetry involving a_{-1} and a_1 is hyponormal .

Proof . $\varphi\left(e^{i\theta}\right)=\sum_{n=-N}^{N} a_n e^{in\theta}$

$a_N e^{iN\theta}=2e^{i5\theta}$

$\bar{a}_N=2e^{-i5\theta}/e^{-iN\theta}$

$\bar{a}_1=2e^{-i5\theta}/e^{-i\theta}$

$\bar{a}_2=2e^{-i5\theta}/e^{i2\theta}$

$\bar{a}_3=2e^{-i5\theta}/e^{-i3\theta}$

$a_{-N}e^{-iN\theta}=e^{-i5\theta}\Rightarrow a_{-N}=e^{-i5\theta}/e^{-iN\theta}$

$a_{-1}=\dfrac{e^{-i5\theta}}{e^{i\theta}},\ a_{-2}=\dfrac{e^{-i5\theta}}{e^{i2\theta}}\ ,a_{-3}=\dfrac{e^{-i5\theta}}{e^{i3\theta}}$

$$2e^{-i5\theta}/e^{-iN\theta}\begin{pmatrix} e^{-i5\theta}/e^{i2\theta} \\ e^{i5\theta}/e^{i3\theta} \\ \vdots \\ e^{-i5\theta}/e^{-iN\theta} \end{pmatrix}=e^{-i5\theta}/e^{-iN\theta}\begin{pmatrix} 2e^{i5\theta}/e^{-i2\theta} \\ 2e^{-i5\theta}/e^{-i3\theta} \\ \vdots \\ 2e^{-i5\theta}/e^{-iN\theta} \end{pmatrix}$$

Corollary 1.19. If $\varphi\left(e^{i\theta}\right)=\sum_{n=-m}^{N} a_n e^{in\theta}$, then T_φ is normal if and only if $m=N$,$|a_{-N}|=|a_N|$and

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-N} \end{pmatrix}=a_{-N}\begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix} \qquad (12)$$

Proof. If $m=N$, $|a_{-m}|=|a_N|$, and let $\det\begin{pmatrix} a_{-(m-k)} & a_{-m} \\ \overline{a_{(N-k)}} & \overline{a_N} \end{pmatrix}=0$ for all $k=1,...,N-1$, then by Theorem 1.15. , T_φ is hyponormal and rank $\left[T_\varphi^*,T_\varphi\right]=N-m=o$; that is T_φ normal . Conversely , if T_φ is normal, then by [

Brown – Halmos [13]] , there are scalars $\alpha, \beta \in \mathbb{C}$ and areal –value $\psi \in L^{\infty}$ such that $T_{\varphi} = \alpha T_{\psi} + \beta 1$. As T_{ψ} is a hermitian Toeplitz operator , the Fourier coefficients of ψ satisfy $\hat{\psi}(n) = \overline{\hat{\psi}(-n)}$ for all n ; in particular $|\alpha||a_N| = |\hat{\psi}(N)| = |\hat{\psi}(-N)| = |\alpha||a_{-N}|$, Showing that $|a_{-N}| = |a_N|$. Thus , N=m and (12) holds .

Remark 1.20. For trigonometric polynomials φ satisfying the assumptions of theorem 1.15. the question of whether or not the Toeplitz operator T_{φ} is hyponormal is completely independent of the values the coefficients $a_0, \cdots, a_{N-m}$ of φ .

Example 1.21. Consider following two trigonometric polynomials:

$$\varphi_1\left(e^{i\theta}\right) = e^{-i2\theta} + e^{i3\theta} + e^{i4\theta}$$

$$\varphi_2\left(e^{i\theta}\right) = e^{-i2\theta} + e^{-i\theta} + e^{i3\theta} + e^{i4\theta}$$

Suggests that φ_2 is less likely than φ_1 to induce ahyponorrmal Toepllitz operator , as φ_2 is " less analytic " in that (conanalytic) term $e^{-i\theta}$ in present is φ_2 but not in φ_1 However the opposite is true : Theorem 1.15. shows that T_{φ_2} is hyponormal (with rank -2 selfcommutator) where as T_{φ_1} is not .

Theorem 1.22. Suppose that

$\varphi\left(e^{i\theta}\right) \sum_{n=-N}^{N} a_n e^{in\theta}$, where $N \geq 2$, $|a_N| \neq 0$ and the coefficients of φ satisfy

$$\overline{a_N} \begin{pmatrix} a_{-2} \\ a_{-3} \\ \vdots \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \overline{a_2} \\ \overline{a_3} \\ \vdots \\ \vdots \\ \overline{a_n} \end{pmatrix} . \qquad (13)$$

Then T_{φ} is hyponoormal if only if

$$|a_N|^2 - |a_{-N}|^2 \geq \sqrt{\left|\det\begin{pmatrix} a_{-1} & a_{-N} \\ \overline{a}_1 & \overline{a}_N \end{pmatrix}\right|^2 + d^2 - d} \quad , \qquad (14)$$

Where $d = \frac{1}{2}\left(1 - |a_{-N}|^2 |a_N|^{-2}\right) \sum_{n=2}^{N-1} |a_n|^2$ and d is taken to be o of $N = -2$.

Proof. Assume (14) holds; we are to prove that T_φ is hyponormal . Solving (4) under the condition (13) produces the analytic polynomial $k_p(z)=c_0+c_{N-1}z^{N-1}$,

where $c_0 = a_{-N}/\overline{a_N}$ and $$c_{N-1} = \left(\overline{a_N}\right)^{-2} \det\begin{pmatrix} a_{-1} & a_{-N} \\ \\ \overline{a}_1 & \overline{a_N} \end{pmatrix}.$$

(14) implies that

$$1-|c_0|^2 \geq \sqrt{|c_{N-1}|^2 + d^2\,|a_N|^{-4}} - d\,|a_N|^{-2} \qquad (15)$$

The right-hand side of (19) nonnegative and so $|c_0|\leq 1$.

Now if $|c_0| = 1$, then $|c_{N-1}|=0$ and T_φ is normal ; assume ,therefore , that $|C_0|<1$. Let $k\in H^2$ be function with Fourier series expansion

$$k = k_p\left(e^{i\theta}\right)+c_{N-1}\sum_{n=1}^{\infty}(-1)^n\left(\frac{c_{N-1}\,\overline{c}_0}{|c_{N-1}|}\right)^n e^{i(N-1)(n+1)\theta}.$$

As $\hat{k}(n)=c_n$ for $n=0,\cdots,N-1$, it remains only to prove that k is in the unit boll of H^∞ .

Let $\alpha = \frac{\overline{c_{N-1}\ c_0}}{|c_{N-1}|}$, which is a complex number of modules $|\alpha| = |c_0| <1$ then

$$k(z)=c_0+c_{N-1}z^{N-1}+c_{N-1}\sum_{n=1}^{\infty}(-1)^n\left(\frac{c_{N-1}\,\overline{c}_0}{|c_{N-1}|}\right)^n\left(Z^{N-1}\right)^{n+1}$$

$$=\frac{c_{N-1}}{1-|c_0|^2}\left(\frac{c_0\,\overline{c_{N-1}}}{|c_{N-1}|}+\left(1-|c_0|^2 Z^{N-1}\right)+\left(1-\frac{|c_{N-1}|}{1-|c_0|^2}\right)c_0\right.$$

$$\left.+\frac{c_{N-1}}{1-|c_0|^2}\sum_{n=1}^{\infty}\left(-\frac{\overline{c_{N-1}}\,c_0}{|c_{N-1}|}\right)^n\left(1-|c_0|^2\right)\left(z^{N-1}\right)^{n+1}\right.$$

$$=\frac{c_{N-1}}{1-|\alpha|^2}\left(-\alpha+\sum_{n=0}^{\infty}\alpha^{-n}\left(1-|\alpha|^2\right)\left(z^{N-1}\right)\left(z^{N-1}\right)^{n+1}\right)\left(1-\frac{|c_{N-1}|}{1-|\alpha|^2}\right)c_0$$

$$=\frac{c_{N-1}}{1-|\alpha|^2}\left(\frac{z^{N-1}-\alpha}{1-\overline{\alpha}\,z^{N-1}}\right)+\left(1-\frac{|c_{N-1}|}{1-|\alpha|^2}\right)c_0$$

Because the function $w\to(w-\alpha)(1-\overline{\alpha}\,w)^{-1}$ is a linear fractional transformation, mapping $\mathbf{T}$ onto itself, we obtain the estimate

$$\|k\|_\infty \leq \frac{|c_{N-1}|}{1-|\alpha|^2}\left|\frac{z^{N-1}-\alpha}{1-\overline{\alpha}\,z^{N-1}}\right|+\left(1-\frac{|c_{N-1}|}{1-|\alpha|^2}\right)|c_0|\leq\frac{|c_{N-1}|}{1-|\alpha|^2}+\left(1-\frac{|c_{N-1}|}{1-|\alpha|^2}\right)=1$$

Which proves that $k\in\mathcal{E}(\varphi)$.

Conversely, suppose now that T_φ is hyponormal with respect to the orthonormal basis $\{ z^n : n = 0, 1, \cdots \}$ of H^2, the selfcommutator of T_φ is a matrix with (μ, v) -entry given by $\alpha_{\mu v} = \sum_{j=0}^{\infty} \left(\overline{a_{j-\mu} \, a_{j-v}} - a_{\mu-j} \, \overline{a_{v-j}} \right)$, where $\mu, v = 0, 1, 2, \ldots$

Thus in particular,

$$\alpha_{00} = \sum_{n=1}^{N} \left(|a_n|^2 - |a_{-n}|^2 \right)$$

$$\alpha_{N-1N-1} = |a_N|^2 - |a_{-N}|^2$$

$$\alpha_{0N-1} = \overline{\alpha_{N-1}} = \overline{a_{N a_1}} - a_{-N} \, \bar{a}_1 .$$

The operator $[T_\varphi^*, T_\varphi]$ is positive and, therefore, so is its 2×2 principal submatrix

$$\begin{pmatrix} \alpha_{00} & \alpha_{0N-1} \\ \alpha_{N-1} & \alpha_{N-1N-1} \end{pmatrix}$$

Hence α_{00} and α_{N-1N-1} are nonnegative and

$$0 \le \det \begin{pmatrix} \alpha_{00} & \alpha_{0N-1} \\ \alpha_{N-1} & \alpha_{N-1N-1} \end{pmatrix} = \alpha_{00} \; \alpha_{N-1N-1} - \left| \alpha_{0N-1} \right|^2$$

$$= \left(\sum_{n=1}^{N} \left(|a_n|^2 - |a_{-n}|^2 \right) \right) \left(|a_{-N}|^2 - |a_{-N}|^2 \right) - \left| \overline{a_N} \; a_1 - a_{-N} \, \bar{a}_{-1} \right|^2 ,$$

The symmetry condition (17) Yields $|a_{-n}| = |a_{-N} / a_N| \, |a_n|$ for $n = 2, \ldots, N-1$.
Direct computation reveals that

$$\left(|a_1|^2 - |a_{-1}|^2 \right) \left(|a_N|^2 - |a_{-N}|^2 \right) + \left| \overline{a_N} \; a_{-1} - a_{-N} \, a_1 \right|^2 = \left| \overline{a_N} \, a_1 - a_{-N} \, \overline{a_{-1}} \right|^2 ,$$

and so

$$0 \le \det \begin{pmatrix} \alpha_{00} & \alpha_{0N-1} \\ \alpha_{N-1} & \alpha_{N-1N-1} \end{pmatrix}$$

$$\le \left(|a_N|^2 - |a_{-N}|^2 \right)^2 + \left(|a_1|^2 - |a_{-1}|^2 \right) \left(|a_N|^2 - |a_{-N}|^2 \right)$$

$$- \left| \overline{a_N} \; a_1 - a_{-N} \, \overline{a_{-1}} \right|^2 + \left(|a_N|^2 - |a_{-N}|^2 \right) \sum_{n=2}^{N-1} \left(|a_n|^2 - |a_{-n}|^2 \right)$$

$$= \left(|a_N|^2 - |a_{-N}|^2 \right)^2 - \left| \overline{a_N} \, a_{-1} - a_{-N} \, \overline{a_1} \right|^2$$

$$+ \left(|a_N|^2 - |a_{-N}|^2 \right) \left(1 - \left| \frac{a_{-N}}{a_N} \right|^2 \right) \sum_{n=2}^{N-1} |a_n|^2 .$$

Therefore,

$$|a_N|^2 - |a_{-N}|^2 \geq \sqrt{\left|\det\begin{pmatrix} a_{-1} & a_{-N} \\ \overline{a}_1 & \overline{a}_N \end{pmatrix}\right|^2 + d^2} - d \quad ,$$

Where $d = \frac{1}{2}\left(1 - |a_{-N}|^2 |a_N|^{-2}\right)\sum_{n=2}^{N-1} |a_n|^2$.

Corollary 1.23. If $\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n e^{in\theta}$ is such that

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix} , \qquad (16)$$

then T_φ is hyponormal if and only if $|a_{-N}| \leq |a_N|$

Theorem 1.24. $\varphi \in PC$ If and only if it is right continuous and has both a left – and right – hand limit at every point.

There are certain algebraic relations among Toeplitz operators whose symbols come from these classes including $T_\psi T_\varphi - T_{\psi\varphi} \in K\left(H^2\right)$ for every

$$\varphi \in H^\infty(\mathbf{T}) + C(\mathbf{T}) \text{ and } \psi \in L^\infty(\mathbf{T}) , \qquad (17)$$

And the commutator $\left[T_\varphi , T_\psi\right]$ is compact for every

$$\varphi, \psi \in PC . \qquad (18)$$

Now we add to these relations the following one .

Lemma 1.25. If T_φ is aToeplitz operator with quasicontinuous symbol φ , and if f is an analytic function on an open set containing $\sigma\left(T_\varphi\right)$, then $T_{fo\varphi} - f\left(T_\varphi\right)$ is a compact operator.

Proof . Assume that $\varphi \in QC$. Recall from [8,] that if $\psi \in H^\infty + C(\mathbf{T})$, then T_ψ is Fredholm if and only if ψ is invertible in $H^\infty + C(\mathbf{T})$. Therefore for every $\lambda \notin \sigma\left(T_\varphi\right)$, both $\varphi - \lambda$ and $\overline{\varphi - \lambda}$ are invertible in $H^\infty + C(\mathbf{T})$; hence , $(\varphi - \lambda)^{-1} \in QC$. Using this fact together with (21) we have for $\psi \in L^\infty$ and $\lambda, \mu \in \mathbb{C}$.

$T_{\varphi-\mu} \; T_\psi T_{(\varphi-\lambda)^{-1}} - T_{(\varphi-\mu)\psi(\varphi-\lambda)^{-1}} \in K\left(H^2\right)$, whenever $\lambda \notin \sigma\left(T_\varphi\right)$.

The argument above extend to rational functions to yield: if r is any rational function with all of its poles outside of $\sigma\left(T_\varphi\right)$, then $r\left(T_\varphi\right) - T_{ro\varphi} \in K\left(H^2\right)$.

Suppose f is an analytic function on an open set containing $\sigma\left(T_{\varphi}\right)$. By Runge's theorem there exists a sequence of rational functions r_n such that the poles of r_n lie outside of $\sigma\left(T_{\varphi}\right)$ and $r_n \to f$ uniformaly on $\sigma\left(T_{\varphi}\right)$. Thus $r_n\left(T_{\varphi}\right) \to f\left(T_{\varphi}\right)$ in the norm- topology of $L\left(H^2\right)$. Furthermore because $r_n o\varphi \to fo\varphi$ uniformaly , we have $T_{r_n o\varphi} \to T_{fo\varphi}$ in the norm – topology . Hence , $T_{fo\varphi} - f\left(T_{\varphi}\right) = \lim\left(T_{r_n o\varphi} - r_n\left(T_{\varphi}\right)\right)$, which is compact .

Lemma (1.25) dose not extent to piecewise continuous symbols $\varphi \in Pc$, as we can not guarantee that $T_{\varphi}^n - T_{\varphi^n} \in K\left(H^2\right)$ for each $n \in Z^+$. For example , if $\varphi\left(e^{i\theta}\right) = \chi\mathbf{T}_+ - \chi\mathbf{T}_-$, where $\chi\mathbf{T}_+$ and $\chi\mathbf{T}_-$ are characteristic , functions of , receptively , the upper semicircle and the lower semicircle , then $T_{\varphi}^2 - I$ is not compact .

Corollary 1.26. If T_{φ} is Toeplitz operator with quasicontinuous symbol φ , then for every analytic function f on an open set containing $\sigma\left(T_{\varphi}\right)$,

(i) $w\left(f\left(T_{\varphi}\right)\right) = \sigma\left(T_{fo\varphi}\right)$, and

(ii) $f\left(T_{\varphi}\right)$ is essentially normal and is unitarily equivalent to a compact perturbation of $f\left(T_{\varphi}\right) \oplus M_{fo\varphi}$, where $M_{fo\varphi}$ is the operator of multiplication by $f o\varphi$ on $L^2\left(\mathbf{T}\right)$.

Proof. Because the Weyl spectrum is stable under the compact perturbations , if follows from Lemma (1.25) that

$w\left(f\left(T_{\varphi}\right)\right) = w\left(T_{fo\varphi}\right) = \sigma\left(T_{fo\varphi}\right)$, which proves statement (i) . To prove (ii) , observe that because QC is a closed algebra , the composition of the analytic function f with $\varphi \in QC$ produces a quasicontinuous function $f o\varphi \in QC$. Moreover , by (21) , every Toeplitz operator with quaasicontinuous symbol is essentially normal the (normal) Laurent operator $M_{fo\varphi}$ on $L^2\left(\mathbf{T}\right)$ has its spectrum contained with the spectrum of the (essentially normal) Toeplitz operator $T_{fo\varphi}$. Thus there is the following relationship involving the essentially normal operators $f\left(T_{\varphi}\right)$and $M_{fo\varphi} \oplus f\left(T_{\varphi}\right)$:

$\sigma_e\left(f\left(T_{\varphi}\right) \oplus N_{fo\varphi}\right) = \sigma_e\left(f\left(T_{\varphi}\right)\right)$ and $SP\left(f\left(T_{\varphi}\right)\right) = Sp\left(f\left(T_{\varphi}\right) \oplus M_{fo\varphi}\right)$, where $SP(T)$ denotes the spectral picture , of an operator T . (The spectral picture $SP(T)$is the structure consisting , of the set $\sigma_e\left(T\right)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the Fredholm indices associated with these holes and pseudo holes .) Thus it follows from the Brown –Douglas – Fill more [23] that $f(T_\varphi)$ is compalent to $f(T_\varphi)\oplus M_{fo\varphi}$, in the sense that there exists a unitary operator w and a compact k such that

$w\left(f(T_\varphi)\oplus M_{fo\varphi}\right)w^* + K = f(T_\varphi)$.

Remark 1.27. Corollary (1.26) (i) saying that $\sigma\left(f(T_\varphi)\right)\setminus\sigma(T_{fo\varphi})$ consist of holes with winding number zero.

Theorem 1.28. If in aBanach algebra $A,[a_i]_i$ is asequence of elements commuting with $a\in A$ and such that $a_i\to a$, then $\lim\sigma(a_i)=\sigma(a)$.

The following Lemma is application of above Theorem .

Lemma 1.29 If $\{T_n\}_n$ is a sequence of operators convering to an operator T and such that $[T_n,T]$ is a compact for each n , then $\lim\sigma_e(T_n)=\sigma_e(T)$.

Proof . From Theorem (1.28) anf f π denotes the canonical homomorpism of $L(H)$ onto the Calkin algebra $L(H)/K(H)$, then the assumption give that $\pi(T_n)\to\pi(T)$ and $[\pi(T_n),\pi(T)]=0$ for each n . Hence $\lim\sigma(\pi(T_n))=\sigma(\pi(T))$; that is , $\lim\sigma_e(T_n)=\sigma_e(T)$.

Remark 1.30. Because $T_n\to T$ by the upper – semi continuity of the spectrum , there is apositive integer N such that $\sigma(T_n)\subseteq V$ whenever $n>N$. and V is an open set containing $\sigma_e(T)$.

Theorem 1.31. Suppose $T,T_n\in L(H)$, for $n\in Z^+$, are such that T_n converges to T . Suppose f is any analytic function whose domain is an open set V containing $\sigma(T)$. If $[T_n,T]\in K(H)$ for each n, then

$$\lim w(f(T_n))=w(f(T)) \qquad (19)$$

Proof . If T_n and T commute modulo the compact operators then , whenever T_n^{-1} and T^{-1} exist , T_n,T,T_n^{-1} and T^{-1} all commute modulo the compact operators .Therefore $r(T_n)$ and $r(T)$ also commute modulo $K(H)$ whenever r is a rational function with no poles in $\sigma(T)$ and n is sufficiently large . Using Runge's theorem we approximate f on compact subsets of V by rational functions r_i who poles lie off of V . So there exists a sequence of rational functions r_i whose poles line outside of V and $r_i\to f$ uniformly on compact subsets of V . If $n>N$, then by the function calculus ,

$$f(T_n)f(T)-f(T)f(T_n)=\lim_i\left(r_i(T_n)r_i(T)-r_i(T)r_i(T_n)\right),$$

Which is compact for each n. Furthermore ,

$$\| f(T_n)-f(T)\|=\left\| \frac{1}{2\pi i}\int_\Gamma f(\lambda)\left((\lambda-T_n)^{-1}-(\lambda-T)^{-1}\right)d\lambda\right\| ,$$

$$\leq\frac{1}{2\pi i}\,\ell(\Gamma)\max_{\lambda\in\Gamma}|f(\lambda)|\,.\max_{\lambda\in\Gamma}\left\|(\lambda-T_n)^{-1}-(\lambda-T)^{-1}\right\|,$$

Where Γ is the boundary of V and $\ell(\Gamma)$ denote the are length of Γ. The right – hand side of the above inequality converges to 0, and so $f(T_n)\to f(T)$. By Lemma (1.29) ,

$$Lim\,\sigma_e(f(T_n))=\sigma_e(f(T)).$$

The argument used by J.B . Con way and B .B Morel in [5] used here to obtan the conclusion $\lim w(f(T_n))=(f(T))$.

We now prve the following theorem .

Theorem 1.32: The restriction of σ to the manifold L of all Toeplitz operators is contrnuous at every Toeplitz operator with quasi continuous symbol . Moreover , if $\varphi\in QC,\varphi_n\in L^\infty$, and $\|T_{\varphi_n}-T_\varphi\|\to 0$, then $\lim\ w(f(T_{\varphi_n}))=\sigma(T_{f\,o\varphi})$.

Proof. suppose $\varphi\in QC\,,\varphi_n\in L^\infty$ and $\|T_{\varphi_n}-T_\varphi\|\to 0$. Then by (21) , $[T_{\varphi_n},T_\varphi]\in K(H^2)$. Therefore by theorem (1.2.45) $\lim w(T_{\varphi_n})=w(T_\varphi)$, and hence $\lim\sigma(T_{\varphi_n})=(T_\varphi)$.

Also, because $f\,o\varphi\in QC$ and $f\,o\varphi_n\to f\,o\varphi$, if follows from lemma (1.2.39) that $Lim\ w(f(T_{\varphi_n}))=\lim\sigma(T_{f\,o\varphi_n})=\sigma(T_{f\,o\varphi})$.

The argument of theorem 1.32. is limited to quasi continuous symbols , as we need is ensure that $[T_{\varphi_n},T_\varphi]$ is compact for every n.

Corollary 1.33.The restriction of σ to L_{PC} is continuous , where L_{PC} is the set of all Toeplitz operators having symbols that are uniform limits of piecewise continuous functions.

With a piecewise continuous function φ, we can obtain a continuous curve $\varphi^\#$ by joining $\varphi(e^{i\theta-0})$ and $\varphi(e^{i\theta})(0\leq\theta<2\pi)$ by the line segment $[\varphi(e^{i\theta-0}),\varphi(e^{i\theta})]$.

Theorem 1.34. For every $\varphi\in PC,\sigma_e(T_\varphi)=\varphi^\#(\mathbf{T})$ and $\sigma(T_\varphi)$ consists of $\varphi^\#(\mathbf{T})$ together with some of its holes .

Remark 1.35. observes that the results which developing Toeplitz operators with piecewise continuous symbols in fact hold , more generally , for symbols $\varphi \in L^{\infty}(\mathbf{T})$ having the property that

$$V_{\lambda_o}(\varphi) = \bigcap_{\in > 0} cl\left[\varphi(\lambda_o - \in, \lambda_o + \in)\right] \qquad (20)$$

Is contained in same line segment L_{λ_o} for each $\lambda_o \in$ in this case ,

$$\sigma_e(T_\varphi) = \bigcup_{\lambda \in} conv V_{\lambda_o}(\varphi) \qquad (21)$$

Definition 1.36. The function φ which satisfying (20) call Douglas function ; let $D(\mathbf{T})$ denote the set of all Douglas functions in $L^{\infty}(\mathbf{T})$.

Definition 1.37. Let $G: L^{\infty} \to C(\tilde{\partial} H^{\infty})$ denote the Gelfand transform , where $\tilde{\partial} H^{\infty}$ is the $\breve{S}ilov$ boundary of $H^{\infty}(\mathbf{T})(i.e., \hat{\partial} H^{\infty}$ is the maximal ideal space of $L^{\infty})$. If $\varphi \in L^{\infty}$, then by the Gelfand theory , $\hat{\varphi}(\tilde{\partial} H^{\infty})$ is the spectrum of φ , as an element of L^{∞} , namely , $\hat{\varphi}(\tilde{\partial} H^{\infty})$ is the closure of the essential range ess-ran φ of φ . Now given $\varphi \in L^{\infty}(\mathbf{T})$, let $V_{\lambda_o}(\varphi)$be as in (20) . If φ has the property that ∂ conv $V_{\lambda_o}(\varphi) \subseteq \hat{\varphi}(\tilde{\partial} H^{\infty})$, or that ∂ conv $V_{\lambda_o}(\varphi)$ is contained in some line segment L_{λ_o} for each $\lambda_o \in \mathbf{T}$, then φ will be called pseudo – piecewise continuous . Write PPC for the set of all pseudo – piecewise continuous functions in L^{∞} .

For every $\lambda_o \in \mathbf{T}$ and $\varphi \in D(\mathbf{T})$, conv $V_{\lambda_o}(\varphi) = \partial conv V_{\lambda_o}(\varphi)$, and so $D(\mathbf{T}) \subseteq PPC$ .If $\varphi \in PPC$, then (21) (together with the fact that T_φ is not a Fredholm operator whenever φ cannot be inverted in $L^{\infty}(\mathbf{T})$ gives

$$\bigcup_{\lambda_o \in T} \partial conv V_{\lambda_o}(\varphi) \subseteq \sigma_e(T_\varphi) \qquad (22)$$

The following example shows that the inclusion $D(\mathbf{T}) \subseteq PPC$ is proper .

Example 1.38. There exists $\varphi \in L^{\infty}(\mathbf{T})$ such that $\varphi \in PPc \setminus D(\mathbf{T})$.

Proof. Set

$$\varphi\left(e^{i\theta}\right)=\begin{cases} e^{i\pi\left(1+\frac{1}{2}\sin\frac{1}{\theta}\right)} & \left(0<\theta<\frac{2}{3\pi}\right) \\ \left(\frac{1}{\pi}+i\right)+\frac{1}{\pi}e^{i\frac{3\pi^2}{6\pi^2-8}\left(2\pi-\frac{2}{\pi}-\theta\right)} & \left(\frac{2}{3\pi}\leq\theta\leq 2\pi-\frac{2}{\pi}\right) \\ 2\pi-\theta+i\sin\frac{1}{2\pi-\theta} & \left(2\pi-\frac{2}{\pi}<\theta<2\pi\right) \end{cases}$$

At $\lambda_o=0$, the graphs of $\varphi\left(\mathbf{T}\right)$ and $V_o\left(\varphi\right)$ can by shown . Therefore $conv\, V_o\left(\varphi\right)$ is contained in no line segment and hence $\varphi\in D\left(\mathbf{T}\right)$ But evidently $\partial ConvV_{\lambda_o}\left(\varphi\right)$ for each $\lambda_o\in$. In fact ,

$$\bigcup_{\lambda_o\in\mathbf{T}} V_{\lambda_o}\left(\varphi\right)=\left\{\hat{\varphi}(\gamma):\gamma\in\tilde{\partial}H^{\infty}\right\}$$

Therefore $\varphi\in PPC$.

Theorem 1.39.If $\varphi\in L^{\infty}\left(\mathbf{T}\right)$ and c is a rectifiable simple closed curve lying in $\mathbb{C}|\,\sigma_e\left(T_\varphi\right)$ then $Conv\,\varphi\left(\mathbf{T}\right)$ lies either entirely inside entirely outside of c .

Definition 1.40. The map $\partial\sigma:L\left(H\right)\to k$ sends every operator $T\in L\left(H\right)$ to the topological boundary $\partial\sigma\left(\mathbf{T}\right)$ of its spectrum $\sigma\left(\mathbf{T}\right)$.

Theorem 1.41. The restriction of $\partial\sigma$ to the set of all Tpoelitz operator with pseudo-piecewise continuous symbol is lower-semicontinuous at each Toeplitz operator with Douglas symbol ; that is , if $\varphi_n\in PPc,\varphi\in D\left(\mathbf{T}\right)$ and $\left\|T_{\varphi_n}-T_\varphi\right\|\to 0$ then $\partial\sigma\left(T_\varphi\right)\subseteq\liminf\partial\sigma\left(T_{\varphi_n}\right)$.

Proof . observe that $\liminf\sigma\left(T_{\varphi_n}\right)=\sigma\left(\liminf\sigma\left(T_{\varphi_n}\right)\right)$. Since $Lim\inf\sigma\left(T_{\varphi_n}\right)$. $\subseteq\sigma\left(T_\varphi\right)$ and hence $\operatorname{int}\left(\liminf\sigma\left(T_{\varphi_n}\right)\right)\subseteq\operatorname{int}\sigma\left(T_\varphi\right)$, it suffices to show that $\partial\sigma\left(T_\varphi\right)\subseteq\liminf\sigma\left(T_{\varphi_n}\right)$. Assume $\lambda\notin\liminf\sigma\left(T_{\varphi_n}\right)$. Then there exists a neighborhood $N_1\left(\lambda\right)$ of λ such that doesnot intersect in finitely many $\sigma\left(T_{\varphi_n}\right)$. Thus we choose a subsequence $\left\{\varphi_{n_i}\right\}$ of $\left\{\varphi_n\right\}$ such that $T_{\varphi_{n_i}}-\mu$ is in veritable for each $\mu\in N_1\left(\lambda\right)$, which says that $\varphi_{n_i}\left(\mathbf{T}\right)\cap N_1\left(\lambda\right)=\emptyset$ for each n_i. Since $\left\|\varphi_n-\varphi\right\|_\infty=\left\|T_{\varphi_n}-T_\varphi\right\|\to 0$, there exists a neighborhood $N_2\left(\lambda\right)$, which says that $\varphi\left(\mathbf{T}\right)\cap N_2\left(\lambda\right)=\emptyset$ and $N_2\left(\lambda\right)\subseteq N_1\left(\lambda\right)$. There are two cases to consider.

Case (i) : suppose $\varphi(\mathbf{T})$ winds around $N_2(\lambda)$. By Theorem 1.34. , but since by (25) , $N_2(\lambda) \subseteq \mathbb{C}/\sigma_\ell(T_\varphi)$, it follows that either $N_2(\lambda) \cap \sigma_\ell(T_\varphi) = \emptyset$. Therefore $\lambda \notin \partial\sigma(T_\varphi)$

Case (ii) : Suppose $\varphi(\mathbf{T})$ does not wind around $N_2(\lambda)$. We now claim that $\lambda \notin \bigcup_{\lambda_o \in \mathbf{T}} Conv\, V_{\lambda_o}(\varphi)$

On the contrary , we assume that $\lambda \in ConvV_{\lambda_o}(\varphi)$ for some $\lambda_o \in \mathbf{T}$. Since $\varphi(\mathbf{T}) \cap N_2(\lambda) = \emptyset$, and $\varphi \in D(\mathbf{T}), \lambda$ must lie in some line segment $L_{\lambda o}(\varphi)$ such that $L_{\lambda_o}(\varphi) \cap \varphi(\mathbf{T}) \neq \emptyset$. Since $\|\varphi_{n_i} - \varphi\| \to 0$, we have $V_{\lambda_o}(\varphi_{n_i}) \to V_{\lambda_o}(\varphi)$ and hence $\partial ConvV_{\lambda_o}(\varphi_{n_1}) \to \partial conV_{\lambda_o}(\varphi)$.
But since $\partial conV_{\lambda_o}(\varphi)$ is contained in a line segment and , by (26) , $\partial convV_{\lambda_o}(\varphi_{n_i}) \subseteq \sigma_e(T_{\varphi_{n_i}})$, it follows that for each neighborhood $N(\lambda)$, there exists $a\, \mu \in N(\lambda)$ such that $T_{\varphi_{n_i}} - \mu$ is not Fredholm , which gives a contradictio. Therefore $\lambda \notin \bigcup_{\lambda_o \in \mathbf{T}} Conv\, V_{\lambda_o}(\varphi)$. Thus by (26), $T_\varphi - \lambda$ is Fredholm . Now because for every $T \in L(H)$, $\partial\sigma(T) \backslash \sigma_e(T)$ consists of isolated points of $\sigma(T)$. We conclude $\lambda \notin \partial\sigma(T_\varphi)$ is connected. This complete the proof.
We now have the extension of corollary 1.26 with the following result.
Theorem 1.42.The restriction of σ to the set of all Toeplitz operators with pseudo piecewise continuous symbols is continuous at each Toeplitz operator with Douglas symbl.

Proof . Suppose $\varphi_n \in PPC,\ \varphi \in D(\mathbf{T})$ and $\|T_{\varphi_n} - T_\varphi\| \to 0$. By theorem (1.41)

$\sigma(T_\kappa)^\wedge = \left(\liminf \sigma(T_{\varphi_n})\right)^\wedge$,

Where $\hat{k}$ denotes the polynomial - convex hull of k consequently, the passage from $\liminf\ \sigma(T_{\varphi_n})$ to $\sigma(T_\varphi)$ is the filling of some holes of $\liminf\ \sigma(T_{\varphi_n})$. Thus if $\sigma(T_\varphi)$ has no holes , then clearly $\sigma(T_\varphi) = \liminf \sigma(T_{\varphi_n})$. If $\sigma(T_\varphi)$ has a hole Ω , then $\partial\,\Omega$ can be regarded as a " local closed curve " see [9] determined by conv$V_\lambda(\varphi)$. As $\partial\Omega \subseteq \bigcup_{\lambda_o \in \mathbf{T}} conv\ V_{\lambda_o \in \mathbf{T}}(\varphi) = \bigcup_{\lambda_o} \partial convV_{\lambda_o}(\varphi)$,
we have $\partial\Omega = \bigcup_{\lambda o \in S} \partial convV_{\lambda_o}(\varphi)$ for some subset S of $\mathbf{T}$.

Because also $\bigcup_{\lambda_o\in\mathbf{T}} \partial conv V_{\lambda_o}\left(\varphi_{n_i}\right)\rightarrow\bigcup_{\lambda_o\in\mathbf{T}} \partial Conv\, V_{\lambda_o}\left(\varphi\right)$, we conclude that for sufficiently large n_i, φ_{n_i} be haves like a Douglas function locally on S. Thus the index theory for continuous symbols can be applied for this local closed curve [9].

But $\|\varphi_n-\varphi\|_\infty\rightarrow$ and so for sufficiently large n,
$-ind\left(T_\varphi-\lambda\right)=wn(\varphi-\lambda)=wn(\varphi_n-\lambda)=-ind\left(T_{\varphi_n}-\lambda\right)$ for each $\lambda\in\Omega$. Hence $\sigma(T_\varphi)\|\lim\inf\sigma\left(T_{\varphi_n}\right)$ has no hole with non – zero winding number , and consequently

$$\sigma\left(T_\varphi\right)=Lim\inf\sigma\left(T_{\varphi_n}\right).$$

Now we show Welyl's theorem for analytic functions of toeplitz operator:
Theorem 1.43. weyl's theorem holds for T if
$w(T)=\sigma(T)\setminus\pi_{00}(T)$,
Where $\pi_{oo}(T)$ is the set of isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.
Theorem 1.44. The set of operators for which Weyl's theorem holds includes all semiformal operators and all Toeplitz operators .
Lemma 1.45. suppose that φ is continuous and f is an analytic function defined on some open set containing $\sigma\left(T_\varphi\right)$. Then

$$\sigma\left(T_{f o\varphi}\right)\subseteq F\left(\sigma\left(T_\varphi\right)\right), \qquad (23)$$

and equality occurs if and only if weyl's theorem holds for $f\left(T_\varphi\right)$.
Proof . By corollary (1.2.40) ,
$\sigma\left(T_{fo\varphi}\right)=w\left(f\left(T_\varphi\right)\right)\subseteq\sigma\left(f\left(T_\varphi\right)\right)=f\left(\sigma\left(T_\varphi\right)\right)$. Because $\sigma(T)$ is connected , so is $f\left(\sigma\left(T_\varphi\right)\right)=\sigma\left(f\left(T_\varphi\right)\right)$; therefore the set $\pi_{00}\left(f\left(T_\varphi\right)\right)$ is empty . Again by corollary (1.2.40) , $w\left(f\left(T_\varphi\right)\right)=\sigma\left(T_{fo\varphi}\right)$ and so
$w\left(f\left(T_\varphi\right)\right)=\sigma\left(f\left(T_\varphi\right)\right)\setminus\pi_{00}\left(f\left(T_\varphi\right)\right)$ If and only if $\sigma\left(T_{fo\varphi}\right)=f\left(\sigma\left(T_\varphi\right)\right)$.
Remark 1.46. If φ is not continuous , it is possible for weyl'e theorem to hold for some $f\left(T_\varphi\right)$ without $\sigma\left(T_{fo\varphi}\right)$ being equal to $f\left(\sigma\left(T_\varphi\right)\right)$. On example is as follows .

Let $\varphi\left(e^{i\theta}\right)=e^{i\theta/3}\ \left(0\leq\theta<2\pi\right)$, a piecewise continuous function. The operator T_φ is invertible but T_{ϕ^2} is not ; hence $0\in\sigma\left(T_{\varphi^2}\right)\setminus\left\{\sigma\left(T_\varphi\right)\right\}^2$. However $w\left(T_\varphi^2\right)=\sigma\left(T_\varphi^2\right)$, and $\pi_{00}\left(T_\varphi^2\right)$ is empty ; Therefore weyl's theorem holds for T_φ^2.

Question 1.47. If T_φ is aToeplitz operators , then does weyl's theorem hold for T_φ^2 ?

Answer: is no .

Note: The answer of 1.47. begin with a spectral property of Toeplitz operators with continuous symbols .

Example 1.48.There exists acontinuous function $\varphi\in C(\mathbf{T})$ such that $\sigma\left(T_{\varphi^2}\right)\neq\left\{\sigma\left(T_\varphi\right)\right\}^2$.

Proof . Let φ be defined by

$$\varphi\left(e^{i\theta}\right)=\begin{cases}-e^{2i\theta}+1 & \left(0\leq 0\leq\pi\right)\\ e^{-2i\theta}-1 & \left(\pi\leq 0\leq 2\pi\right)\end{cases}$$

The orientation of the graph of φ can be shown clearly φ is continuous and , φ has winding number +1 with respect to the hole of c_1; the hole of c_2 has winding number -1 . Thus we have

$\sigma_e\left(T_\varphi\right)=\varphi(\mathbf{T})$ and $\sigma\left(T_\varphi\right)=conv\,\varphi(\mathbf{T})$

On the other hand , straightforward calculation shows that $\varphi^2(\mathbf{T})$ is the cardioid $r=2(1+\cos\theta)$. In particular , $\varphi^2(\mathbf{T})$ traverses the cardioid once in acounterclockwise direction and then traverses the cardioid once in aclock wise direction .

Thus $w_n\left(\varphi^2-\lambda\right)=0$ for each λ in the hole of $\varphi^2(\mathbf{T})$. Hence $T_{\varphi^2-\lambda}$ is aweyl operator and is therefore , invertible for each λ in the hole of $\varphi^2(\mathbf{T})$. This implies that $\sigma\left(T_{\varphi^2}\right)$ is the cardioid $r=2\left(1+\cos\theta\right)$ .But because $\left\{\sigma\left(T_\varphi\right)\right\}^2=\left\{conv\,\varphi(\mathbf{T})\right\}^2=\left\{(r,\theta):r\leq 2(1+\cos\theta)\right\}$, if follows

$\sigma\left(T_{\varphi^2}\right)\neq\left\{\sigma\left(T_\varphi\right)\right\}^2$

Remark 1.49. It is instructive to observe that lemma 1.45. gives a necessary condition for T_φ to be hyponormal . we recall [17] hat if $T\in L(H)$ is hyponormal , then weyl's theorem holds for every $f(T)$. In conjunction with lemma 1.45. , this is to say that if T_φ is hyponormal , then $\sigma\left(T_{fo\varphi}\right)-f\left(\sigma\left(T_\varphi\right)\right)$. But this necessary condition is not sufficient , for a slight extension of theorem [1] , [17] which show that weyl's theorem holds for $f\left(T_\varphi\right)$, where T_φ is the cohyponormal Toeplitz operator with symbol $\varphi\left(e^{i\theta}\right)=e^{i\theta}$; hence $\sigma\left(T_{fo\varphi}\right)=f\left(\sigma\left(T_\varphi\right)\right)$.

We conclude by studying continuous symbols φ that have the property that weyl's theorem holds for $f(T_\varphi)$, for every analytic function f on a neighourhood of $\sigma(T_\varphi)$.

Theorem 1.50. If $\varphi \in C(\mathbf{T})$ is such that $\sigma(T_\varphi)$ has planar Lebesgue measure zero, then $\sigma(T_{fo\varphi}) = f(\sigma(T_\varphi))$ for every analytic function f defined on an open set containing $\sigma(T_\varphi)$.

Proof. As φ is continuous , so is $f o \varphi$ and thus $\sigma_e(T_\varphi) = \varphi(\mathbf{T})$ and $\sigma_e(f(T_\varphi)) = \sigma_e(T_{fo\varphi}) = f o \varphi(\mathbf{T})$. The planar measure of $\sigma(T_\varphi)$ is zero ; because $\sigma(T_\varphi)$ is a compact connected set consisting of $\varphi(\mathbf{T})$ and some of its holes , it follows that $\partial\sigma(T_\varphi) = \sigma_e(T_\varphi) = \sigma(T_\varphi)$, which is just a continuous curve . Furthermore, as analytic functions map open connected sets on to open connected open sets , we have that $\partial\sigma(f(T_\varphi)) = \sigma_e(f(T_\varphi)) = \sigma(f(T_\varphi))$. Thus $\sigma(f(T_\varphi)) \subseteq \sigma(T_{fo\varphi})$, which together with (23) gives the result.

Remark 1.51.We note here that Toeplitz operators whose symbol satisfies the hypothesis of theorem 1.50 are essentially normal of type " normal + compact " . To see this let D be a diagonal operator whose spectrum is $\varphi(\mathbf{T})$. Because T_φ and D are both essentially normal and $SP(T_\varphi) = SP(D)$, it follows from the Brown – Douglas – Fillmore theorem that T_φ and D are complent ; that is , $T_\varphi = N + K$ for some normal operator N on H^2 and some $K \in K(H^2)$. this observation is of interest because if $\sigma(T_\varphi)$ has planar Lebesgue measure zero and , further , if T_φ is hyponormal , then by putnam's inequality T_φ is normal and $\varphi(\mathbf{T})$ must be a line segment .

Theorem 1.52. If the winding number of $\varphi \in C(\mathbf{T})$ with respect to each hole of $\varphi(\mathbf{T})$ is nonnegative (or is nonpositive) , then $\sigma(T_{fo\varphi}) = f(\sigma(T_\varphi))$ for every analytic function defined on an open set containing $\sigma(T_\varphi)$.

Proof . suppose that the holes of $\varphi(\mathbf{T})$ have only nonnegative winding numbers . Since φ is continuous , it follows that $\sigma_e(T_\varphi) = \varphi(T)$ and

$$\sigma_e(T_{fo\varphi}) = \sigma_e(f(T_\varphi)) = f(\sigma_e(T_\varphi)) \qquad (24)$$

If $\varphi(\mathbf{T})$ has no holes or has holes of winding number zero only , then $\sigma(T_\varphi) = \sigma_e(T_\varphi)$; thus

$f\left(\sigma\left(T_{\varphi}\right)\right)=f\left(\sigma_e\left(T_{\varphi}\right)\right)=\sigma_e\left(f\left(T_{\varphi}\right)\right)=\sigma_e\left(T_{fo\varphi}\right)\subseteq\sigma\left(T_{fo\varphi}\right)$,

Which together with (23) gives $\sigma\left(T_{fo\varphi}\right)=f\left(\sigma\left(T_{\varphi}\right)\right)$.

Now assume that there exists at least a hole Ω of $\varphi(\mathbf{T})$ such that $wn(\varphi-\lambda)\neq 0$ for all $\lambda\in\Omega$ .Namely, $wn(\varphi-\lambda)=w>0$, for all $\lambda\in\Omega$.In view of (24) , it sufficient to show that

$f\left(\sigma\left(T_{\varphi}\right)\right)\setminus f\left(\sigma_e\left(T_{\varphi}\right)\right)\subseteq\sigma\left(T_{fo\varphi}\right)\setminus\sigma_e\left(T_{fo\varphi}\right)$. Thus the proof is completed by showing that if $\lambda\in\Omega$, then $f(\lambda)\in\sigma\left(T_{fo\varphi}\right)$. Suppose that $\lambda\in\Omega$; thus $T_{\varphi}-\lambda$ is Fredholm with $ind\left(T_{\varphi}-\lambda\right)=-wn(\varphi-\lambda)=-w<0$. Write

$f(z)-f(\lambda)=(z-\lambda)(z-\mu_1)^{\alpha_1}\dots(z-\mu_n)^{\alpha_n}F(z)$,

where $\alpha_i\in Z^+,\mu_i\in\sigma\left(T_{\varphi}\right)(1\leq i\leq n)$ and $F(z)$ is analytic and has no zeros in $\sigma\left(T_{\varphi}\right)$. We have

$fo\varphi-f(\lambda)=(\varphi-\lambda)(\varphi-\mu_1)^{\alpha_1}\dots(\varphi-\mu_n)^{\alpha_n}Fo\varphi$.

From (24) , $T_{fo\varphi-f(\lambda)}$ is Fredholm and hence $fo\varphi-f(\lambda)$ is invertible on . So each $\varphi-\mu_1(1\leq i\leq n)$ and $Fo\varphi$ vanish nowhere on T . Therefore $T_{\varphi-\mu_1}$ and $T_{Fo\varphi}$ are all Fredholm . By assumption , $wn(\varphi-\mu_i)\geq 0$, and because $Fo\varphi$ has no zeros in $\sigma\left(T_{\varphi}\right)$, $wn(Fo\varphi)=0$. Thus

$$ind\left(T_{Fo\varphi-f(\lambda)}\right)=wn\left\{(\varphi-\lambda)(\varphi-\mu_1)^{\alpha^1}\dots(\varphi-\mu_n)^{\alpha_n}F(\varphi)\right\}$$

$$=-wn(\varphi-\lambda)-\sum_{i=1}^{n}\alpha_i\, wn(\varphi-\mu_i)<0$$,

Which shows that $T_{fo\varphi-f(\lambda)}$ is not Awelye operator and hence is not invertible. We conclude that $f(\lambda)\in\sigma\left(T_{fo\varphi}\right)$. The proof of case of non positive winding numbers is similar

Example (1.2.67) [4] : If φ is of the form $P\left(\frac{a}{z}+bz\right)$,where $a,b\in R$ and P is any polynomial , then

$\sigma\left(T_{fo\varphi}\right)=f\left(\sigma\left(T_{\varphi}\right)\right)$.

Proof. if $a=b$, then T_{φ} is hermitian and the desired conclusion is clear . If $a\neq b$, set $\psi=\frac{a}{z}+bz$. Then

$$\psi(\mathbf{T})=\left\{(u,v)\in\mathbb{C}:\left(\frac{u}{b+a}\right)^2+\left(\frac{v}{b-a}\right)^2=1\right\}$$,

Which is a circle or an ellipse . Thus $\varphi(\mathbf{T})=(p\,o\,\psi)(\mathbf{T})=p(\psi)(\mathbf{T})$, which has no holes has exactly one hole (because polynomials map continuous curves onto conclusion curves and open sets onto open sets) . The conclusion nows follows from Theorem 1.52. .

Remark 1.53. Lemma 1.45. and theorem 1.50, 1.52 hold for quasicontinuous symbol φ . In this case , if T_φ is Fredholm , then the index of T_φ is the negative of the winding number with respect to the origin of the curve $\hat{\varphi}\left(r e^{i\theta}\right)$ for $1-\delta< r <1$, and

$$\sigma_e\left(T_\varphi\right)=\bigcap_{0<\delta<1} cl\left\{\hat{\varphi}\left(r\, e^{i\theta}\right):1-\delta<r<1\right\},$$

Where $\hat{\varphi}$ is the harmonic extension of φ to the open unit disk D [8].

Remark 1.55.The index of a hyponormal operator is always nonpositive and therefore, in general , the holes of the essential spectrum of a hyponormal operator cannot have negative winding numbers . This fact may lead one to believe that if $\varphi(\mathbf{T})$ has no hole with negative winding number (in particular , in case that φ is atrigonometric polynomial . Then T_φ is hyponormal . But such is not the case . For example , if

$\varphi_1\left(e^{i\theta}\right)=e^{-2i\theta}+e^{i\theta}+e^{2i\theta}$, and $\varphi_2\left(e^{i\theta}\right)=e^{-2i\theta}-e^{-i\theta}+e^{i\theta}+e^{2i\theta}$, then $\varphi_1(\mathbf{T})$ has just one essential hole whose winding number is +1 , and $\varphi_2(\mathbf{T})$ has no hole . as shown in figure (4) . But the theorem (1.41) , T_{φ_1} and T_{φ_2} both fail to be hyponormal .

Remark 1.56. Recall [23] that an operator $T\in L(H)$ is quasitriangular if there exists an increasing sequence $\{P_n\}$ of projections of finite rank in $L(H)$ that converges strongly to the identity and satisfies $\left\|P_n\, T\, P_n - T\, P_n\right\|\to 0$. By work of A postal , Foias, and Voiculescu , it is known that T is quasitriangular if and only if $S\,P(T)$ contains no hole or pseudohole with negative winding number . Rewrite theorem 1.52. as follows . If T_φ is quasitriangular (or T_φ^* is a quasitriangular) Toeplitz operator with continuous symbol φ , then $\sigma\left(T_{fo\varphi}\right)=f\left(\sigma\left(T_\varphi\right)\right)$. In Remark 1.54. we showed that even if T_φ^* is a quasitriangular operator (with trigonometric polynomial symbol φ) , T_φ may fail to be hyponormal . In spite of this , it would be interesting to have amethod by which one could determine the winding number of curves given by trigonometric polynomials with respect to the various holes these polynomial produce . We expect the solution will make extensive use of Theorem 1.52.

Spectrum of Toeplitz operators with Harmonic polynomial symbols on Bergman space

In the following Theorem we investigate the structure of the spectrum of the Toeplitz operator $T_{\underline{z}+p}$ via certain analytic properties:

Theorem 2.1(Pearcy): Let T be a bounded linear operator on a Hilbert space H and H be " a hole in $\sigma_e(\ T)$ " (which is a bounded component of $C \setminus \sigma_e(\ T\ )$) such that

Index $(T - \lambda I) = 0\ , \lambda \in H$,

Then either

(a) $H \cap \sigma(\ T) = \emptyset$,

(b) $H \cap \sigma(\ T) \subset \emptyset$, or

(c) $H \cap \sigma(\ T)$ is a countable set of isolated eigenvalues of T, each having finite multiplicity. Furthermore, the intersection of $\sigma(\ T)$ with the unbounded component of $C \setminus \sigma_e(\ T\ )$ is a countable set of isolated eigenvalues of T, each of which has finite multiplicity. The following theorem gives a characterization for the eigenvalues of a class of Toeplitz operators with harmonic symbols on the Bergman space, which is useful for us to study the isolated points in the spectra of Toeplitz operators with some bounded harmonic symbols. The following theorem gives a characterization for the eigenvalues of a class of Toeplitz operators with harmonic symbols on the Bergman space, which is useful for us to study the isolated points in the spectra of Toeplitz operators with some bounded harmonic symbols.

Theorem 2.2. Let p be a function in $H^\infty \cap C\left(\underline{D}\right)$ .Suppose that λ is a complex number not in the essential spectrum of the Toeplitz operator $T_{\underline{z}+p}$. Then λ is an eigenvalue of $T_{\underline{z}+p}$ if and only if either $1 + z[p(z) - \lambda]$ dose not vanish on the unit disk or $1 + z[p(z) - \lambda]$ has finitely many simple zeros $\{z_1, \dots, z_k\}$ in the unit disk Which satisfy

$$z_j^2\ \acute{p}\left(z_j\right) = \frac{n_j+2}{n_j+1}\ for\ some\ integer\ n_j \in$$

$\{0,1,2, \dots\}\ with\ j = 1,2, \dots, k$.

The above theorem leads to the following complete characterization on the invertibility of the Toeplitz operator $T_{\underline{z}+p}$ with $p \in H^\infty \cap C\left(\underline{D}\right)$Immediately.

Theorem 2.3. Let p be a function in $H^\infty \cap C\left(\underline{D}\right)$.Then the Toeplitz operator $T_{\underline{z}+p}$ is invertible on the Bergman space L_α^2 if and only if the following two conditions hold:

(i) $1 + zp$ has no zeros on the unit circle ∂D ;

(ii) $1 + zp$ has exactly one simple zero z_0 in the open disk D which satisfies that

$$z_0^2\ \acute{p}(z_0) - \frac{n+2}{n+1} \neq 0$$

For any nonnegative integer n.

Now we show that the spectrum of Toeplitz operator $T_{\underline{z}+p}$ is connected for every quadratic polynomial p .

Theorem 3.3. Let $\varphi(z) = \underline{z} + p(z)$, where p a quadratic polynomial is. The spectrum of the is Toeplitz operator T_φ given by

$\sigma(T_\varphi) = \varphi\,(\partial D) \cup \ \{\lambda \in C : \lambda \notin \varphi(\partial D)\ and\ wind(\varphi(\partial D), \lambda) \neq 0\}$,

Which coincides with the spectrum of the corresponding Hardy-Toeplitz operator with symbol $e^{-i\theta} + p(e^{i\theta})$. Hence the spectrum

Of $T_{\underline{z}+p}$ is connected for every $p(z) = az^2 + bz + c$ with $a, b, c \in C$.

Corollary 3.4. Let $\varphi(z) = \underline{z} + (\ az^2 + bz + c\ )$ where a, b and c are all complex constants. Then the Toeplitz operator T_φ is invertible if and only if

The cubic polynomial $az^3 + bz^2 + cz + 1$ has a unique zero in the unit disk D with multiplicity 1, but does not have any

Zero on the unit circle ∂D.

Corollary 3.5. $\varphi(z) = \underline{z} + (\ az^2 + bz + c\ )\ with\ a, b, c \in C$ .Then we have

$$\sigma(T_\varphi) \subset clos[\varphi(D)].$$

CONCLUSION

This paper establishes conditions under which the Toeplitz operators T_φ is hyponormal with φ is trigonormaltic polynomial

$\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n\ e^{in\varphi}$, where $a_N \neq 0$

If $\varphi\left(e^{i\theta}\right) = \sum_{n=-N}^{N} a_n\, e^{in\theta}$, where $a_N \neq 0$, and if $c_0, c_1, \ldots c_{N-1} \in \not{C}$ are obtained from the coefficients of φ by solving the recurrence relation $c_0 = \frac{a_{-N}}{\bar{a}_N}$

$c_n\left(\ \bar{a}_N\ \right)^{-1}\left(a_{-N+n} - \sum_{j=0}^{n-1} C_j\, \overline{a_{N-n+j}}\right)$, for n=1, … , N-1

Then, $\sum_{j=0}^{N-1} |c_j| \leq 1$

.And with φ is trigonormaltic polynomial

$\varphi\left(e^{i\theta}\right) = \sum_{n=-m}^{N} a_n e^{in\theta}$, then T_φ is normal if and only if $m = N$,$|a_{-N}| = |a_N|$and

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix}$$

.And investigate the structure of the spectral picture of the Toeplitz operators $T_{\underline{z}+p}$ with harmonic symbols on Bergman space .And we have result that If $\varphi(z) = \underline{z} + (\ az^2 + bz + c\ )$ where $a,\ b$ and c are all complex constants. Then the Toeplitz operator T_φ is invertible if and only if the cubic polynomial $az^3 + bz^2 + cz + 1$ has a unique zero in the unit disk D with multiplicity 1, but does not have any Zero on the unit circle ∂D.

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